Wilson-'t Hooft operators in four-dimensional gauge theories and *S*-duality

Anton Kapustin*

California Institute of Technology, Pasadena California 91125 USA

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We study operators in four-dimensional gauge theories which are localized on a straight line, create electric and magnetic flux, and in the UV limit break the conformal invariance in the minimal possible way. We call them Wilson-'t Hooft operators, since in the purely electric case they reduce to the well-known Wilson loops, while in general they may carry 't Hooft magnetic flux. We show that to any such operator one can associate a maximally symmetric boundary condition for gauge fields on $\mathrm{AdS}_E^2 \times S^2$. We show that Wilson-'t Hooft operators are classified by a pair of weights (electric and magnetic) for the gauge group and its magnetic dual, modulo the action of the Weyl group. If the magnetic weight does not belong to the coroot lattice of the gauge group, the corresponding operator is topologically nontrivial (carries nonvanishing 't Hooft magnetic flux). We explain how the spectrum of Wilson-'t Hooft operators transforms under the shift of the θ-angle by 2π. We show that, depending on the gauge group, either $SL(2, \mathbb{Z})$ or one of its congruence subgroups acts in a natural way on the set of Wilson-'t Hooft operators. This can be regarded as evidence for the S-duality of $N = 4$ super-Yang-Mills theory. We also compute the one-point function of the stress-energy tensor in the presence of a Wilson-'t Hooft operator at weak coupling.

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I. INTRODUCTION

It is known to quantum field theory aficionados, but not appreciated widely enough, that local operators need not be defined as local functions of the fields which are used to write down the Lagrangian. The simplest example is an operator which creates a winding state in the theory of a free periodic boson in 2d. Another example is a twist operator for a single Majorana fermion in 2d. The second example shows that it is a matter of convention which field is regarded as fundamental: the massless Majorana fermion can be reinterpreted as the continuum limit of the Ising model at the critical temperature, and then it is more natural to regard one of the twist operators (the one with fermion number 0) as the fundamental object, since it corresponds to the spin operator of the Ising model. Another famous example of this phenomenon is the fermion-boson equivalence in two dimensions. The unifying theme of all these examples is that one can define a local operator by requiring the "fundamental" fields in the path-integral to have a singularity of a prescribed kind at the insertion point. It may happen that the presence of the singularity can be detected from afar for topological reasons. For example, in the presence of the fermionic twist operator, the fermion field changes sign as one goes around the insertion point. In such cases, one can say that the operator insertion creates topological disorder. (In fact, all examples mentioned above fall into this category). But it is important to realize that the idea of defining local operators by means of singularities in the fundamental fields is more general than that of a topological disorder operator.

Local operators not expressible as local functions of the fundamental fields play an important role in various dualities. For example, a certain winding-state operator in the theory of a free periodic boson in 2d is a fermion which satisfies the equations of motion of the massless Thirring model. Thus the massless Thirring model is dual to a free theory. Upon perturbing the Thirring model Lagrangian by a mass term, one finds that it is dual to the sine-Gordon model [1]. The fermion number of the Thirring model corresponds to the soliton charge of the sine-Gordon model.

So far all our examples have been two-dimensional. Already in one of the first papers on the sine-Gordon-Thirring duality it was proposed that similar dualities may exist in higher-dimensional fields theories [2]. More specifically, S. Mandelstam proposed that an Abelian gauge theory in 3d admitting Abrikosov-Nielsen-Olesen (ANO) vortices may have a dual description where the ANO vortex state is created from vacuum by a fundamental field. Much later, such 3d dual pairs have indeed been discovered [3–7]. Duality of this kind is known as 3d mirror symmetry. The operator in the gauge theory creating the ANO vortex has been constructed in Refs. [8,9]. It is a topological disorder operator in the sense that the gauge field looks like the field of Dirac monopole near the insertion point. In other words, the operator is (partially) characterized by the fact that the first Chern class of the gauge bundle on any 2-sphere enclosing the insertion point is nontrivial. In the dual theory, the same operator is a local gauge-invariant function of the fundamental fields.

Analogous dualities exist for certain non-Abelian gauge theories in 3d. These theories do not have any conserved topological charges and therefore do not have topological disorder operators. Nevertheless, they admit local operators characterized by the fact that near the insertion point the gauge field looks like a Goddard-Nuyts-Olive (GNO) monopole [10]. Such operators have been studied in

*Electronic address: kapustin@theory.caltech.edu

Ref. [11], where it was argued that 3d mirror symmetry maps them to ordinary local operators whose vacuum expectation values (VEVs) parametrize the Higgs branch of the dual theory.

It is natural to try to extend these considerations to 4d gauge theories, the primary objective being a better understanding of various conjectural dualities. In 4d a punctured neighborhood of a point is homotopic to S^3, and since vector bundles on S^3 do not have interesting characteristic classes, we do not expect to find any local topological disorder operators in such theories. But as we know from 3d examples, this does not rule out the possibility of a local operator which creates a singularity in the fields at the insertion point.

A more serious problem is that we are mostly interested in operators which can be studied at weak coupling. This means that the singularity in the fields that we allow at the insertion point must be compatible with the classical equations of motion. If we want the operator to have a well-defined scaling dimension, we also have to require that the allowed singularity be scale-invariant. Together these two requirements are too strong to allow nontrivial solutions in four-dimensional field theories. For example, in a theory of a free scalar field an operator insertion at the origin can be thought of as a local modification of the Klein-Gordon equation of the form

$$\Box\phi(x) = A(\phi(x), \partial)\delta^4(x),$$

where A is a polynomial in the field ϕ and derivatives. However, scale-invariance requires A to have dimension -1, which is impossible. Similarly, in four dimensions no local modifications of the Maxwell equations which would preserve scale-invariance are possible. (The situation in two- and three-dimensional field theories is very different in this respect.)

To circumvent this problem, we recall that in some sense the most basic gauge-invariant operators in a gauge theory are not local, but distributed along a line. These are the famous Wilson loop operators [12]. Their importance stems from the fact that they serve as order parameters for confinement: in the confining phase the expectation value of a Wilson loop in a suitable representation exhibits area law [12]. Similarly, to detect the Higgs phase, one makes use of the 't Hooft loop operators [13]. More general phases with "oblique confinement" and mixed Wilson-'t Hooft loop operators also exist.

For line operators, there is no conflict between scale invariance and the equations of motion. Consider, for example, an operator supported on a straight line in $\mathbb{R}^4$. It is clear that if we consider a singularity in the fields which corresponds to a Dirac monopole (embedded in the non-Abelian gauge group), it will satisfy both requirements. More generally, one can consider singularities which correspond to a dyon world line.

Another nice feature of line operators is that in some cases they can be regarded as topological disorder operators. Indeed, the punctured neighborhood of a straight line is homotopic to S^2, and in a non-Abelian gauge theory with gauge group G one may consider gauge bundles which have a nontrivial 't Hooft magnetic flux [a class in $H^2(S^2, \pi_1(G))$] on S^2. By definition, these are 't Hooft loop operators. However, even if the gauge group is simply-connected, one may still consider nontopological line operators which are defined by the requirement that near the insertion line the gauge field looks like the field of a GNO monopole. This is completely analogous to the situation in 3d [11]. We will continue to call such operators 't Hooft operators, even though they may carry trivial 't Hooft magnetic flux. Note also that the line entering the definition of the Wilson and 't Hooft operators can be infinite without boundary. This is why we prefer to use the terms "Wilson operator" and "'t Hooft operator" instead of the more common names "Wilson loop operator" and "'t Hooft loop operator."

In this paper we study general Wilson-'t Hooft operators in 4d gauge theories using the approach of Refs. [8,9,11]. We are especially interested in the action of various dualities on these operators. In this paper we focus on a particular example: S-duality of the $N = 4$ super-Yang-Mills (SYM) theory. As usually formulated, this conjecture says that the $N = 4$ SYM theory with a simple gauge Lie algebra $\mathfrak{g}$ and complexified coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi}{g^2}$ is isomorphic to a similar theory with the Langlands-dual gauge Lie algebra $\hat{\mathfrak{g}}$ and coupling $\hat{\tau} = -1/\tau$ [14–16]. We will focus on a corollary of this conjecture, which says that for a given gauge Lie algebra $\mathfrak{g}$ the self-duality group of $N = 4$ SYM is either $SL(2, \mathbb{Z})$ or one of its congruence subgroups $\Gamma_0(q)$ for $q = 2, 3$, depending on $\mathfrak{g}$ [17,18]. We will refer to this group as the S-duality group. A natural question to ask is how the S-duality group acts on the Wilson loop operator of the theory. For example, in the simply-laced case, where the S-duality group is $SL(2, \mathbb{Z})$, one can ask how the generator S acts on a Wilson loop operator. The usual answer that it is mapped to the 't Hooft loop operator is clearly inadequate, since Wilson loop operators are parametrized by irreducible representations of $\mathfrak{g}$, while according to 't Hooft's original definition [13] 't Hooft loop operators are parametrized by elements of $\pi_1(\hat{G}) = Z(G)$, where G is the gauge group. In this paper we describe the action of the full S-duality group on Wilson-'t Hooft operators. The existence of such an action can be regarded as evidence for the S-duality conjecture.

One can characterize a local operator $\mathcal{O}$ in a conformal field theory by the OPE of $\mathcal{O}$ with conserved currents, for example, the stress-energy tensor $T_{\mu\nu}$. We define similar quantities for line operators and compute them at weak coupling. We observe that they are sensitive to the θ-angle of the theory. This is a manifestation of the Witten effect [19].

The outline of the paper is as follows. In Sec. II we define more precisely the class of line operators we are going to be interested in, as well as some of their quantitative characteristics. In Sec. III we study line operators in free 4d theories. In particular, we discuss the action of $SL(2, \mathbb{Z})$ transformations on general Wilson-'t Hooft operators in the free Maxwell theory. In Sec. IV we classify Wilson-'t Hooft operators in non-Abelian gauge theories and determine the action of the S-duality group on them. We also compute the expectation value of the stress-energy tensor in the presence of a Wilson-'t Hooft operator to leading order in the gauge coupling. In Sec. V we summarize our results and propose directions for future work. In the appendix we collect some standard facts about compact simple Lie algebras.

II. THE DEFINITION OF LINE OPERATORS

Our viewpoint is that a field theory is defined by specifying a flow from a UV fixed point. The UV fixed point is a conformal field theory (CFT), and all local operators should be defined in this CFT. This approach is especially convenient for studying local topological disorder operators, because local operators in a flat-space d-dimensional CFT are in one-to-one correspondence with states in the same CFT on $S^{d-1} \times \mathbb{R}$. This can be seen by performing a Weyl rescaling of the flat Euclidean metric by a factor $1/r^2$, where r is the distance to the insertion point. After Weyl rescaling, the metric becomes the standard metric on $S^{d-1} \times \mathbb{R}$, while the insertion point is at infinity (in the infinite past, if we regard the coordinate $\log r$ parametrizing $\mathbb{R}$ as the Euclidean time).

Similarly, line operators should also be defined in the UV fixed point theory, so from now on we limit ourselves to line operators in CFTs. The main requirement that we impose on a line operator is that its insertion preserves all the space-time symmetries preserved by the line, regarded as a geometric object. This condition is meant to replace the usual requirement that local fields be (quasi)primaries of the conformal group. Of course, a generic line breaks all conformal symmetry, so to get an interesting restriction we will limit ourselves to straight lines (this implicitly includes circular lines, because they are conformally equivalent to straight lines). As in the case of local operators, it is useful (although not strictly necessary) to perform a Weyl rescaling of the metric so that the line is at infinite distance. In cylindrical coordinates, the metric of $\mathbb{R}^d$ has the form

$$ds^2 = dt^2 + dr^2 + r^2 d\Omega_{d-2}^2,$$

where $d\Omega_{d-2}^2$ is the standard metric on a unit $d-2$-dimensional sphere, so it is natural to rescale the metric by a factor $1/r^2$, getting

$$d\tilde{s}^2 = \frac{dt^2 + dr^2}{r^2} + d\Omega_{d-2}^2.$$

This is the product metric on $H^2 \times S^{d-2}$, where H^2 is the Lobachevsky plane, i.e. the upper half-plane of $\mathbb{C}$ with the Poincaré metric. Another name for H^2 is Euclidean AdS^2. The line is at the boundary of H^2. After this Weyl rescaling, it is clear that the subgroup of the conformal group preserved by the line is $SL(2, \mathbb{R}) \times SO(d-1)$.

In this picture, a straight line operator corresponds to a choice of a boundary condition for the path-integral of the CFT on $H^2 \times S^{d-2}$. By assumption, we only consider boundary conditions which preserve the full group of motions of this space. Possible boundary conditions for fields on AdS space have been extensively studied in connection with AdS/CFT correspondence, see e.g. Refs. [20,21], and we will implicitly make use of some of these results later on.

The use of the Weyl rescaling is not mandatory: it is useful only because it makes the action of $SL(2, \mathbb{R}) \times SO(d-1)$ more obvious. Alternatively, one can simply excise the line from $\mathbb{R}^d$ and allow for fields to have singularities on this line compatible with the required symmetries. We will often switch between the flat-space picture and the $H^2 \times S^{d-2}$ picture. We will distinguish fields on $H^2 \times S^{d-2}$ with a tilde; this is necessary, because fields with nonzero scaling dimension transform nontrivially under Weyl rescaling. Specifically, in going from $\mathbb{R}^d$ to $H^2 \times S^{d-2}$ a tensor field of scaling dimension p is multiplied by r^p.

Given a line operator W, one may consider Green's functions of local operators with an insertion of W. We will be especially interested in the normalized 1-point function of the stress-energy tensor in the presence of W. We will denote it

$$\langle T_{\mu\nu}(x)\rangle_W = \frac{\langle W T_{\mu\nu}(x)\rangle}{\langle W\rangle}.$$

We use normalized Green's functions because in most cases of interest W needs multiplicative renormalization, and in normalized Green's functions the corresponding arbitrariness in the definition of W cancels between the numerator and the denominator. Assuming that $T_{\mu\nu}$ is symmetric and traceless, the form of this 1-point function is completely fixed by the $SL(2, \mathbb{R}) \times SO(d-1)$ invariance. For definiteness, from now on we will specialize to 4d CFTs. (There are also interesting line operators in 3d CFTs; we plan to discuss them elsewhere.) Then the 1-point function of $T_{\mu\nu}$ takes the form:

$$\langle T_{00}(x)\rangle_W = \frac{h_W}{r^4}, \qquad \langle T_{ij}(x)\rangle_W = -h_W \frac{\delta_{ij} - 2n_i n_j}{r^4},$$
$$\langle T_{0j}(x)\rangle_W = 0.$$

Here we used the coordinates $x = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$, where $x^0 = t$ and $r = |\vec{x}|$. We also let $n_i = x^i/r$. The coefficient h_W is a number characterizing the line operator W. In some sense it is analogous to the scaling dimension of a local primary operator, so we will call h_W the scaling weight of W. This analogy does not go very far though, because h_W does not seem to admit an interpretation in terms of commutation relations of W with conserved quantities. Indeed, to compute the vacuum expectation value of the commutator of W with a conserved charge corresponding to a conformal Killing vector field ξ^μ, one has to integrate

$$\xi^\mu \langle T_{\mu\nu}(x)\rangle_W$$

over the 2-sphere given by the equation $r = \epsilon$, and then take the limit $\epsilon \to 0$. But it is easy to see that for Killing vector fields corresponding to $SO(3)$ symmetry the integral vanishes identically, while for conformal Killing vector fields corresponding to $SL(2, \mathbb{R})$ symmetry it diverges as $1/\epsilon$. (There is also a divergence due to the infinite length of the straight line). We will also see that the scaling weight need not be positive, or even real.

Note that the above result for the 1-point function of $T_{\mu\nu}$ becomes much more obvious if one uses the $H^2 \times S^2$ picture. There, the expectation value of the tensor $\tilde{T} = \tilde{T}_{\mu\nu} dx^\mu \otimes dx^\nu$ takes the form

$$\langle \tilde{T}\rangle_W = h_W(ds^2(H^2) - ds^2(S^2))$$

It is obvious that this is the most general traceless symmetric tensor on $H^2 \times S^2$ which is invariant under all isometries.

III. LINE OPERATORS IN FREE 4D CFTS

To illustrate our approach to line operators, in this section we will consider some examples in free 4d CFTs. This will also serve as a preparation for studying line operators in $N = 4$ super-Yang-Mills, which is an exactly marginal deformation of a free theory.

A. Free scalar

We begin with the theory of a free scalar in 4d. The flat-space Euclidean action is

$$S = \frac{1}{2g^2}\int d^4x(\partial\phi)^2.$$

This action is invariant under Weyl rescaling if in curved space-time we allow for an extra term in the action $\frac{1}{6}R\phi^2$. The improved stress-energy tensor (which is also the Einstein tensor for the action with the extra term) is

$$T_{\mu\nu} = g^{-2}[\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\delta_{\mu\nu}(\partial\phi)^2 - \tfrac{1}{6}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\phi^2]. \tag{3.1}$$

This CFT admits a line operator

$$V_\lambda = \exp\left(\lambda\int\phi(t,\vec{0})dt\right), \qquad \lambda \in \mathbb{C}. \tag{3.2}$$

It is easy to see that it preserves the required symmetries (basically, this follows from the fact that ϕ is a primary field of dimension 1). Evaluating the 1-point function of $T_{\mu\nu}$ in the usual way, we find the scaling weight of V_λ:

$$h_\lambda = -\frac{\lambda^2 g^2}{96\pi^2} \tag{3.3}$$

In the $H^2 \times S^2$ picture, the boundary condition is the "free" boundary condition. This means that in the limit $r \to 0$ the field $\tilde{\phi} = r\phi$ behaves as follows:

$$\tilde{\phi} = a(t) + \tilde{O}(r), \tag{3.4}$$

where the function a is not constrained. The symbol $\tilde{O}(r^n)$ means "of order r^n in the $H^2 \times S^2$ picture," while $O(r^n)$ will mean "of order r^n in the $\mathbb{R}^4$ picture." Later, when we consider tensor fields on $H^2 \times S^2$ (resp. $\mathbb{R}^4$), we will say that a tensor is $\tilde{O}(r^n)$ [resp. $O(r^n)$] if its components in an orthonormal basis are of order r^n.

If we use the $\mathbb{R}^4$ picture, Eq. (3.4) means that ϕ is allowed to have a singularity of the form

$$\phi = \frac{a}{r} + O(1).$$

In addition, the path-integral contains a factor V_λ (in either picture). Note that this factor, as well as the classical action, are UV divergent for $a \neq 0$. It is convenient to deal with this divergence by restricting the region of integration in the expression for the action to $r \geq \epsilon > 0$ and by regularizing V_λ as follows:

$$V_\lambda(\epsilon) = \exp\left(\frac{1}{4\pi}\lambda\int\phi(t,\epsilon\vec{n})dtd^2\sigma\right).$$

Here $d^2\sigma$ is the area element of a unit 2-sphere parametrized by θ, ϕ, and $\vec{n}$ denotes a unit vector in $\mathbb{R}^3$ pointing in the direction specified by θ, ϕ. In the end we have to send ϵ to zero.

Since we are dealing with a Gaussian theory, the normalized 1-point function of $T_{\mu\nu}$ is simply the value of $T_{\mu\nu}$ on the solution of classical equations of motion. The only solution satisfying the required boundary condition at $r = 0$ and preserving all the symmetries is

$$\phi_{cl} = \frac{a_0}{r}, \qquad a_0 \in \mathbb{R}.$$

Note that in the $H^2 \times S^2$ picture this is simply a constant scalar field: $\tilde{\phi} = a_0$. It obviously preserves the full isometry group of $H^2 \times S^2$, which is one way to explain why V_λ is a good line operator.

The constant a_0 is determined by requiring that the variation of $\log V_\lambda$ to cancel the boundary term in the variation of the classical action. The former is

$$\frac{1}{4\pi\epsilon}\int\lambda\delta a dt d^2\sigma,$$

while the latter is

$$\frac{1}{\epsilon g^2}\int a\delta a dt d^2\sigma.$$

Requiring the equality of the two terms, we find

$$a_0 = \frac{\lambda g^2}{4\pi}.$$

Plugging this scalar background into the classical expression Eq. (3.1) for $T_{\mu\nu}$, we find again the result Eq. (3.3).

B. Free Maxwell theory

Next we consider the free Maxwell theory. The flat-space action is

$$S_0 = \frac{1}{4g^2}\int d^4x F_{\mu\nu}F_{\mu\nu}.$$

The form $A = A_\mu dx^\mu$ has scaling dimension 0, so we do not have to make a distinction between A and $\tilde{A}$, or F and $\tilde{F}$. The stress-energy tensor is

$$T_{\mu\nu} = g^{-2}[-F_{\mu\lambda}F_{\nu\lambda} + \tfrac{1}{4}\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}].$$

The simplest line operator is the Wilson operator

$$W_n = \exp\left(in\int_L A_\mu dx^\mu\right).$$

If the gauge group is compact, then n must be an integer. The operator W_n inserts an infinitely massive particle of charge n whose world line is L. An easy computation gives the scaling weight of W_n:

$$h(W_n) = \frac{n^2 g^2}{32\pi^2}. \tag{3.5}$$

In the $H^2\times S^2$ picture, one considers the path-integral over topologically trivial gauge fields with free boundary conditions for F_{0r} and fixed boundary conditions for the rest of the components of $F = dA$ at $r = 0$:

$$F = dA = (a(t) + O(r))\frac{dr\wedge dt}{r^2} + O(1).$$

where the function $a(t)$ is arbitrary. In the gauge $A_r = 0$, the corresponding boundary condition on A reads

$$A = (a(t) + O(r))\frac{dt}{r} + O(1).$$

One also needs to insert W_n in the path-integral.

The corresponding solution of the classical equations of motion is simply a constant electric field on $H^2\times S^2$:

$$F = a_0\frac{dr\wedge dt}{r^2}.$$

In the $\mathbb{R}^4$ picture, this is simply the Coulomb field of a charged particle, as expected. The magnitude of the electric field is again determined by requiring the cancellation of the boundary variation of S and the variation of W_n. Substituting this classical solution into the classical $T_{\mu\nu}$, one again finds Eq. (3.5). Note that a constant electric field on H^2 is clearly invariant under isometries. This shows that the Wilson operator is a good line operator in our sense.

By electric-magnetic duality, we expect to have a "magnetic" Wilson line parametrized by an integer m. By definition, this is a 't Hooft operator corresponding to an insertion of a Dirac monopole of charge m. In the $H^2\times S^2$ picture, this clearly means that F has the following boundary behavior:

$$F = \frac{m}{2}\mathrm{vol}(S^2) + O(1).$$

If the gauge group is compact, m must be an integer. In the gauge $A_r = 0$, both A_0 and A_i have fixed boundary conditions:

$$A_0 = O(1), \qquad A_i dx^i = \frac{m}{2}(1-\cos\theta)d\phi + O(1).$$

The unique solution of equations of motion satisfying these boundary conditions is

$$F = \frac{m}{2}\mathrm{vol}(S^2),$$

i.e. a constant magnetic field on S^2. It is obviously invariant under all isometries of $H^2\times S^2$. From the $\mathbb{R}^4$ perspective, this is the field of a Dirac monopole of magnetic charge m:

$$F = \frac{m}{4}\frac{\epsilon_{ijk}x^i dx^j\wedge dx^k}{r^3}.$$

Substituting this solution into the classical $T_{\mu\nu}$, we find the scaling weight of the 't Hooft operator:

$$h(H_m) = \frac{m^2}{8g^2}. \tag{3.6}$$

This result is exact, because we are dealing with a Gaussian theory. Note that $h(W_n)$ at gauge coupling g^2 is equal to $h(H_n)$ at gauge coupling $\hat{g}^2 = \frac{4\pi^2}{g^2}$. This follows from the S-duality of the Maxwell theory on $\mathbb{R}^4$: the inversion of the gauge coupling

$$\tau = \frac{2\pi i}{g^2} \mapsto \hat{\tau} = -\frac{1}{\tau}$$

leaves the theory invariant and maps W_n to H_n.

Next we generalize this discussion in two directions. First of all, we will consider "mixed" Wilson-'t Hooft operators corresponding to the insertion of an infinitely massive dyon with an electric charge n and magnetic charge m. Second, we will allow for a nonvanishing θ-angle.

We impose the following boundary conditions on the gauge fields on $H^2\times S^2$:

$$F = (a(t) + O(r))\frac{dr\wedge dt}{r^2} + \frac{m}{2}vol(S^2) + O(1).$$

That is, asymptotically we have a fixed magnetic field on S^2 and an electric field of an unconstrained magnitude. We also insert a factor W_n in the path-integral. The action is now

$$S_\theta = S_0 - \frac{i\theta}{8\pi^2}\int F \wedge F.$$

Since the θ-term is a total derivative, its variation is a boundary term, and therefore affects the value of a_0. Indeed, the boundary variation of the full action is now

$$\frac{4\pi}{g^2}\frac{1}{\epsilon}\int a\delta a dt - \frac{im\theta}{2\pi}\frac{1}{\epsilon}\int \delta a dt$$

while the variation of W_n is the same as before:

$$\delta \log W_n = in\frac{1}{\epsilon}\int \delta a dt.$$

Requiring their equality, we get

$$a_0 = \frac{ig^2}{4\pi}\left(n + \frac{m\theta}{2\pi}\right).$$

Evaluating $T_{\mu\nu}$ on the corresponding classical background, we find the scaling weight of the Wilson-'t Hooft operator:

$$h(WH_{(n,m)}) = \frac{g^2}{32\pi^2}\left[\left(n + \frac{m\theta}{2\pi}\right)^2 + \frac{4\pi^2 m^2}{g^2}\right].$$

Introducing the complexified gauge coupling

$$\tau = \frac{2\pi i}{g^2} + \frac{\theta}{2\pi},$$

we can rewrite this in a manifestly $SL(2,\mathbb{Z})$-covariant way:

$$h(WH_{(n,m)}) = \frac{1}{16\pi}\frac{|n + m\tau|^2}{\mathrm{Im}\tau}.$$

The S and T transformations act as follows:

$$S\colon \tau \mapsto -1/\tau, \qquad (n, m) \mapsto (-m, n), \tag{3.7}$$

$$T\colon \tau \mapsto \tau + 1, \qquad (n, m) \mapsto (n + m, m). \tag{3.8}$$

C. BPS Wilson-'t Hooft operators in $N = 4$ Maxwell theory

Let us now consider $N = 4$ supersymmetric Maxwell theory. This is a product of the free Maxwell theory, four copies of the free fermion theory, and six copies of the free scalar theory. In such a theory it is natural to consider Bogomol'nyi-Prasad-Sommerfield (BPS) line operators, i.e. line operators which preserve some supersymmetry. We can take the following ansatz for such an operator:

$$WH^{\mathrm{BPS}}_{(n,m)} = W_n H_m V_\lambda,$$

where V_λ is the line operator Eq. (3.2) for one of the six scalar fields. The coefficient λ is fixed by the BPS requirement. Note that it is incorrect, in general, to fix λ by requiring the expression

$$W_n V_\lambda = \exp\left(\int (inA_0 + \lambda\phi)dt\right)$$

to be preserved by some supersymmetries. This would ignore the possibility that the SUSY variation of the action produces a boundary term, which has to be canceled by the SUSY variation of $W_n V_\lambda$. To find the right value of λ, one can either analyze in detail these boundary terms, or, more simply, require that the corresponding solution of the classical equations of motion be BPS. The relevant solution of the equations of motion is

$$F = \frac{i}{2}\frac{\mathrm{Re}(n + m\tau)}{\mathrm{Im}\tau}\frac{dt \wedge dr}{r^2} + \frac{m}{2}\mathrm{vol}(S^2), \qquad \phi = \frac{\lambda g^2}{4\pi r}.$$

This field configuration is a BPS dyon if and only if

$$\lambda = |n + m\tau|.$$

Then the scaling weight of the BPS Wilson-'t Hooft operator is

$$h(WH^{\mathrm{BPS}}_{(n,m)}) = \frac{1}{24\pi}\frac{|n + m\tau|^2}{\mathrm{Im}\tau}.$$

This expression is exact because the theory is Gaussian.

IV. WILSON-'T HOOFT OPERATORS IN NON-ABELIAN 4D GAUGE THEORIES

In this section we study Wilson-'t Hooft operators in conformally-invariant non-Abelian gauge theories in 4d. Some of our statements apply to any such theory, while others apply only to the most well-known example, the $N = 4$ super-Yang-Mills theory. The gauge group, to be denoted G, is assumed to be simple and compact. Its Lie algebra will be denoted $\mathfrak{g}$.

A. Wilson operators

Wilson operators in a theory with a gauge group G are classified by irreducible representations of G and have the form

$$W_R = \mathrm{Tr}_R P \exp\left(i\int A_0 dt\right),$$

where Tr_R is the trace in the irreducible representation R. In the $H^2 \times S^2$ picture and in the gauge $A_r = 0$, one imposes the following boundary conditions:

$$A_0 = \frac{a(t)}{r} + O(1), \qquad A_i = O(1),$$

where $a(t)$ is an arbitrary function valued in the Lie algebra $\mathfrak{g}$. That is, the boundary conditions for A_0 are free, while the boundary conditions for A_i are fixed. One also has to insert the operator W_R in the path-integral.

Evaluating the expectation value of the stress-energy tensor in the presence of the Wilson line at weak coupling, one easily finds the leading-order result for the scaling weight:

$$h(W_R) = \frac{g^2 c_2(R)}{64\pi^2} + O(g^4), \tag{4.1}$$

where c_2 is the second Casimir of the representation R.

B. 't Hooft operators and GNO monopoles

Generalizing the results of Sec. III, we expect that 't Hooft operators correspond to fixed boundary conditions for all the fields. We also expect A_0 to be of order $O(1)$ at $r = 0$. Together with the requirement of $SL(2, \mathbb{R}) \times SO(3)$ invariance, this completely fixes the boundary behavior of $F = dA + A \wedge A$:

$$F = \frac{B}{2}\mathrm{vol}(S^2) + O(1), \tag{4.2}$$

where B is a section of the adjoint bundle on S^2. The section B does not have to be constant (such a constraint would not be gauge-invariant anyway), but if we want this asymptotics to respect the $SO(3)$ isometry, then B must be covariantly constant. It was shown in Ref. [10] that this implies a quantization law for B:

$$\exp(2\pi i B) = id_G. \tag{4.3}$$

One can always use gauge transformations to make B at any chosen point on S^2 to belong to some fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. If the adjoint bundle is trivial, we can regard B as a constant valued in $\mathfrak{t}$ and satisfying Eq. (4.3). If the adjoint bundle is nontrivial, we can cover S^2 with two charts (by removing either the south or the north pole), and in each of the charts we can choose a trivialization where B is a $\mathfrak{t}$-valued constant. In such a trivialization, the gauge field asymptotics has the form

$$A_\mu dx^\mu = \frac{B}{2}(1 - \cos\theta)d\phi + O(1),$$

where we only wrote down the expression in the chart covering the north pole ($\theta = 0$). This is simply a Dirac monopole embedded into the non-Abelian group G.

Note that Goddard *et al.* regarded Eq. (4.2) as describing the asymptotics of the gauge field at $r \to \infty$, while we regard it as describing the asymptotics at $r \to 0$. Thus although the mathematical manipulations are the same as in Ref. [10], the physical interpretation is different.

Let us first specialize to the case when the gauge group G has a trivial center. That is,

$$G = G_0 := \tilde{G}/Z(\tilde{G}),$$

where $\tilde{G}$ is the unique simply connected compact Lie group with Lie algebra $\mathfrak{g}$. This case is of particular interest to us, because in the $N = 4$ super-Yang-Mills theory all the fields are in the adjoint representations of $\mathfrak{g}$, and it is natural to take G_0 as the gauge group. Note that the action of $N = 4$ super-Yang-Mills depends only on $\mathfrak{g}$, not on the Lie group G, thus the choice of G is up to us. In flat space-time, the only effect of taking a different G with the same Lie algebra $\mathfrak{g}$ would be to restrict the allowed values of the 't Hooft magnetic flux for line operators. Since we would like to classify all possible line operators, we choose not to put any unnecessary restrictions on the magnetic flux.

If $G = G_0$, then the condition Eq. (4.3) is equivalent to the following quantization law [10]:

$$\alpha(B) \in \mathbb{Z}, \qquad \forall \alpha \in \Phi. \tag{4.4}$$

Here $\Phi \subset \mathfrak{t}^*$ is the set of roots of $\mathfrak{g}$. Our Lie algebra conventions are summarized in the appendix. We find it convenient to keep the normalization of the Killing metric on $\mathfrak{g}$ undetermined until as late as possible. Therefore we do not identify $\mathfrak{t}$ and $\mathfrak{t}^*$, and regard roots and weights as elements of $\mathfrak{t}^*$, while coroots H_α are regarded as elements of $\mathfrak{t}$.

The element $B \in \mathfrak{t}$ is defined only modulo the action of the Weyl group [10]:

$$w_\alpha : B \mapsto B' = B - \alpha(B)H_\alpha, \qquad \alpha \in \Phi.$$

Since $\beta(H_\alpha)$ is integer for any two roots α, β, the quantization condition is obviously invariant under the action of the Weyl group.

Let us pick an arbitrary Ad-invariant metric on $\mathfrak{g}$. The Langlands-dual Lie algebra $\hat{\mathfrak{g}}$ is defined as follows. Its Cartan subalgebra is $\hat{\mathfrak{t}} = \mathfrak{t}^*$, and the set of roots $\hat{\Phi} \subset \mathfrak{t}$ is defined to be the set of coroots of $\mathfrak{g}$. Then the set of coroots of $\hat{\mathfrak{g}}$ coincides with the set of roots of $\mathfrak{g}$. The Weyl groups of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ are the same. Then B can be regarded as an element of $\hat{\mathfrak{t}}^*$ which takes integer values on any coroot of $\hat{\mathfrak{g}}$. This is precisely the definition of a weight of $\hat{\mathfrak{g}}$. Thus specifying the equivalence class of B under the action of the Weyl group which satisfies the quantization law Eq. (4.4) is the same as specifying a dominant weight of $\hat{\mathfrak{g}}$. The latter is the same as specifying an irreducible representation of $\hat{\mathfrak{g}}$. This observation was made for the first time in Ref. [10], where for this reason elements of $\mathfrak{t}$ satisfying Eq. (4.4) were called magnetic weights. Magnetic weights form a lattice in $\mathfrak{t}$ which contains the lattice spanned by all coroots H_α (see appendix).

We conclude that 't Hooft operators in a gauge theory with a centerless simple compact Lie group G are classified by irreducible representations of $\hat{\mathfrak{g}}$, or, equivalently, by orbits of magnetic weights in $\mathfrak{t}$ under the action of the Weyl group of $\mathfrak{g}$. We will denote the 't Hooft operator corresponding to a magnetic weight B by H_B. This is a finer classification than the classification of 't Hooft operators by their 't Hooft magnetic flux. Indeed, the latter takes values in $\pi_1(G_0) = Z(\tilde{G})$, which can be identified with the quotient

$$\Lambda_{\mathrm{mw}}/\Lambda_{\mathrm{cr}}$$

where $\Lambda_{\mathrm{mw}} \in \mathfrak{t}$ is lattice of magnetic weights, and $\Lambda_{\mathrm{cr}} \in \mathfrak{t}$ is the coroot lattice. The 't Hooft magnetic flux of a line operator H_B can be identified with the image of $B \in \Lambda_{\mathrm{mw}}$ under the projection

$$\Lambda_{\mathrm{mw}} \to \Lambda_{\mathrm{mw}}/\Lambda_{\mathrm{cr}}.$$

If the gauge theory in question contains representations of $\mathfrak{g}$ which transform nontrivially under $Z(\tilde{G})$ [e.g. if $\mathfrak{g} = \mathfrak{sl}_N$ and there are fields in the fundamental representation of $SU(N)$], then the gauge group is some cover of G_0. Let $\Gamma \subset \mathfrak{t}$ denote the kernel of the exponential mapping

$$\exp: \mathfrak{t} \to G, \qquad B \exp(2\pi i B).$$

The set Γ is a lattice in $\mathfrak{t}$, and the condition Eq. (4.3) says that $B \in \Gamma$. The lattice Γ is contained in Λ_{mw} and contains Λ_{cr}. The center of G is actually the quotient $\Lambda_{\mathrm{mw}}/\Gamma$, while the fundamental group $\pi_1(G)$ is the quotient $\Gamma/\Lambda_{\mathrm{cr}}$. Thus if the center of the gauge group G is nontrivial, one gets an extra constraint on the 't Hooft magnetic flux of 't Hooft operators: the flux must lie in $\Gamma/\Lambda_{\mathrm{cr}}$, which is a subgroup of $\Lambda_{\mathrm{mw}}/\Lambda_{\mathrm{cr}}$. The lattice Γ can also be thought of as the weight lattice of some compact Lie group $\hat{G}$ with Lie algebra $\hat{\mathfrak{g}}$. Namely, $\hat{G}$ is the group whose center is $\Gamma/\Lambda_{\mathrm{cr}}$. This group was introduced in Ref. [10], where it was proposed that a gauge theory with gauge group G may admit a dual description as a gauge theory with gauge group $\hat{G}$. For this reason it is usually called the GNO-dual of G in the physics literature. In the mathematical literature $\hat{G}$ is called the Langlands-dual of G because of its role in the Langlands program.

C. Classification of Wilson-'t Hooft operators

Now we consider mixed Wilson-'t Hooft operators. A fundamental observation made in Refs. [22–24] (see also Ref. [25]) is that in the presence of a non-Abelian magnetic field global gauge transformations are restricted to those which preserve the Lie algebra element B. Thus we expect Wilson-'t Hooft operators to be labeled by a pair (B, R), where $B \in \mathfrak{t}$ is a magnetic weight, and R is an irreducible representation of the stabilizer subgroup of B.

Before showing that this is indeed the case, we need to address one possible source of confusion. When considering 't Hooft operators, it was natural to take G to have the smallest possible center. In particular, for $N = 4$ super-Yang-Mills it was natural to take G to have a trivial center. On the other hand, when considering Wilson operators in the same theory, it is natural to allow arbitrary irreducible representations of $\tilde{G}$, the universal covering group of G_0. In general, a representation of $\tilde{G}$ is only a projective representation of G_0, so one may wonder if R in the pair (B, R) must be a representation of the stabilizer of B in G_0 or in $\tilde{G}$. The latter is the correct answer. The simplest way to see this is to note that in the case $B = 0$ it gives the correct result (that Wilson operators are classified by representations of $\tilde{G}$). Here is another way to argue the same thing. Basically, we are saying that if the dynamical fields all transform trivially under $Z(\tilde{G})$, then it makes sense to probe the theory with a source which transforms in an arbitrary representation of $Z(\tilde{G})$ and carries an arbitrary 't Hooft magnetic flux in $\pi_1(G_0)$. This is familiar from the Abelian case: if there is a single dyon in the universe, and no other electrically or magnetically charged particles, then the Dirac-Zwanziger-Schwinger condition is vacuous, and the electric and magnetic charges are completely arbitrary. Nontrivial conditions arise only when we consider a pair of dyons. In the present context, we expect a constraint on pairs of allowed Wilson-'t Hooft operators arising from the requirement of locality.

Coming back to the issue of classification of Wilson-'t Hooft operators, we will impose the following (fixed) boundary conditions on A_i:

$$A_i dx^i = \frac{B}{2}(1 - \cos\theta) d\phi + O(1),$$

where B is a covariantly constant section of the adjoint bundle on $S^2 \times \mathbb{R}$. This implies that the "spatial" components of F have the following boundary behavior:

$$\frac{1}{2} F_{ij} dx^i dx^j = \frac{B}{4} \frac{\epsilon_{ijk} x^i dx^j \wedge dx^k}{r^3} + O(1).$$

As for A_0, we attempt to impose the free boundary condition, i.e.

$$A_0 = \frac{a}{r} + O(1),$$

where a is an arbitrary section of the adjoint bundle on $S^2 \times \mathbb{R}$. We have to check whether these boundary conditions preserve $SL(2, \mathbb{R}) \times SO(3)$ invariance. The only nontrivial constraint comes from requiring invariance with respect to translations of $t = x^0$:

$$D_0 F_{ij} = 0. \tag{4.5}$$

In the Abelian case, this condition required B to be time-independent. In the present case, this condition first of all implies that we can choose a local trivialization of the adjoint bundle on $S^2 \times \mathbb{R}$ by two charts of the form $U_\pm \times \mathbb{R}$, where U_+, U_- is the standard covering of S^2, so that B is independent of t. In such a trivialization, the condition Eq. (4.5) becomes

$$[A_0, B] = O(r).$$

In particular, the section a should commute with the section B.

We have learned that in the neighborhood of $r = 0$ the gauge field A_0 takes value in the centralizer of B in $\mathfrak{g}$. We will denote this centralizer $\mathfrak{g}_B$. The gauge group $\tilde{G}$ is also broken down to $\tilde{G}_B$, the stabilizer of B in the adjoint representation of $\tilde{G}$. The Lie algebra of $\tilde{G}_B$ is $\mathfrak{g}_B$. Thus we can construct a line operator by picking an irreducible representation R of $\tilde{G}_B$ and inserting in the path-integral the factor

$$W_R = \mathrm{Tr}_R P \exp\left(i \int A_0(t, 0) dt\right). \tag{4.6}$$

We conclude that Wilson-'t Hooft operators are classified by a pair (B, R), where B is an equivalence class of a magnetic weight, and R is an irreducible representation of $\tilde{G}_B$, the stabilizer of B.

In the next subsection we will discuss the action of the S-duality group on the set of Wilson-'t Hooft operators. As a preparation, let us repackage the data (B, R) in a more suggestive form. The group $\tilde{G}_B$ is a compact connected reductive Lie group whose Lie algebra we have denoted $\mathfrak{g}_B$. The Cartan subalgebra of $\mathfrak{g}_B$ can be identified with $\mathfrak{t}$. Let Φ_B be the set of roots of $\mathfrak{g}_B$. It is easy to see that Φ_B is a subset of Φ defined by the condition

$$\alpha(B) = 0. \tag{4.7}$$

Further, the Weyl group $\mathcal{W}_B$ of $\mathfrak{g}_B$ is a subgroup of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ and consists of reflections associated to roots in Φ_B. In other words, $\mathcal{W}_B$ is the stabilizer subgroup of B in $\mathcal{W}$. The maximal tori of $\tilde{G}_B$ and $\tilde{G}$ coincide, so a representation of $\tilde{G}_B$ can be specified by specifying how the maximal torus of $\tilde{G}$ acts. This can be done by picking a weight of $\mathfrak{g}$. If we were discussing representations of $\tilde{G}$, we would also identify weights lying in the same $\mathcal{W}$-orbit. But since we are interested in representations of $\tilde{G}_B$, we only need to identify weights lying in the same $\mathcal{W}_B$ orbit.

To summarize, specifying a $\mathcal{W}$-orbit of a magnetic weight $B \in \mathfrak{t}$ and an irreducible representation R of $\tilde{G}_B$ is the same as specifying a $\mathcal{W}$-orbit of $B \in \mathfrak{t}$ and a $\mathcal{W}_B$ orbit of an ordinary ("electric") weight $\mu \in \mathfrak{t}^*$. Here $\mathcal{W}_B$ is the subgroup of $\mathcal{W}$ leaving B invariant. But clearly this is the same as specifying a pair of weights, one magnetic and one electric, and identifying pairs related by the action of $\mathcal{W}$. Thus Wilson-'t Hooft operators are classified by an equivalence class of pairs (B, μ) under the action of $\mathcal{W}$, where $B \in \mathfrak{t}$ is a magnetic weight, and $\mu \in \mathfrak{t}^*$ is an ordinary weight of $\mathfrak{g}$.

D. *S*-duality

Let us now discuss the action of the S-duality group on the set of Wilson-'t Hooft operators. Recall that for $N = 4$ super-Yang-Mills theory with a simply-laced simple Lie algebra $\mathfrak{g}$ the duality group is conjectured to be $SL(2, \mathbb{Z})$, while for nonsimply-laced Lie algebras it is a subgroup of $SL(2, \mathbb{Z})$ denoted $\Gamma_0(q)$, where $q = 2$ for $\mathfrak{g} = \mathsf{so}, \mathsf{sp}, F_4$ and $q = 3$ for $\mathfrak{g} = G_2$ (see Refs. [17,18]). We remind that the group $\Gamma_0(q)$ is a subgroup of $SL(2, \mathbb{Z})$ consisting of the matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1,$$
$$c \equiv 0 \qquad \mathrm{mod} q.$$

The group $SL(2, \mathbb{Z})$ is generated by three elements S, T, C where $C = -1$ is central and the following relations hold:

$$C^2 = 1, \qquad S^2 = C, \qquad (ST)^3 = C. \tag{4.8}$$

The group $\Gamma_0(q)$ is generated by C, T and ST^qS.

The element C is represented by charge-conjugation. It reverses the sign of the Yang-Mills potential A and curvature F, so it acts on Wilson-'t Hooft operators by

$$C: (\mu, B) \mapsto (-\mu, -B).$$

(If the group G admits only self-conjugate representations, C is a trivial operation. In that case, -1 is in the Weyl group, and multiplying (μ, B) by -1 does not affect the equivalence class of the pair (μ, B).)

The element T corresponds to the shift of the θ-angle by 2π. At this stage it is important to fix a normalization of the invariant metric on $\mathfrak{g}$. We want the coefficient of θ in the action to take value 1 on the minimal instanton. An instanton with a minimal action is obtained by embedding the usual $SU(2)$ instanton into an $SU(2)$ subgroup of $\tilde{G}$ associated with a short coroot. For such an instanton, the topological charge is

$$\tfrac{1}{2}\langle H_\alpha, H_\alpha \rangle.$$

Hence we choose the metric so that short coroots have length $\sqrt{2}$. Having fixed the metric, we define the field theory action to be

$$S = \frac{1}{2g^2} \int \langle F_{\mu\nu}, F_{\mu\nu} \rangle - \frac{i\theta}{8\pi^2} \int \langle F, \wedge F \rangle.$$

In the case of $G = SU(N)$ this normalization of g^2 and θ is equivalent to the following standard one:

$$S = \frac{1}{2g^2} \int \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) - \frac{i\theta}{8\pi^2} \int \mathrm{Tr}(F \wedge F),$$

where Tr is the trace in the fundamental representation.

Having fixed a metric on $\mathfrak{g}$, we get a natural isomorphism $\ell: \mathfrak{t} \to \mathfrak{t}^*$. For any element $a \in \mathfrak{t}$ we let $a^* = \ell(a) \in \mathfrak{t}^*$. Similarly, for any $\rho \in \mathfrak{t}^*$ we let $\rho^* = \ell^{-1}(\rho)$ be the corresponding element of $\mathfrak{t}$. We claim that the T-transformation acts as follows:

$$T: (\mu, B) \mapsto (\mu + B^*, B). \tag{4.9}$$

This makes sense, because for any coroot $H_\alpha \in \mathfrak{t}$ we have, by Eq. (A1),

$$B^*(H_\alpha) = H_\alpha^*(B) = \tfrac{1}{2}\langle H_\alpha, H_\alpha \rangle \alpha(B).$$

Since short coroots have been normalized to satisfy

$$\langle H_\alpha, H_\alpha \rangle = 2,$$

and the length-squared of long coroots is an integral multiple of that for short coroots, we see that $B^*(H_\alpha)$ is necessarily an integer, and therefore $\mu + B^*$ belongs to the weight lattice. It is also easy to see that the map Eq. (4.9) commutes with the action of the Weyl group $\mathcal{W}$, so we get a well-defined map on the set of orbits of $\mathcal{W}$.

Now let us show that shifting the θ-angle by 2π induces the map Eq. (4.9) on Wilson-'t Hooft operators. Recall that $\mu \in \mathfrak{t}^*$ specifies a representation R_μ of $\tilde{G}_B$. The reductive Lie algebra $\mathfrak{g}_B$ is a direct sum of a semisimple Lie algebra $\mathfrak{g}_{ss}$ and an Abelian Lie algebra $\mathfrak{g}_{ab} = \mathfrak{t}_{ab}$. The transformation Eq. (4.9) modifies only the action of $\mathfrak{g}_{ab}$. Indeed, for any root $\alpha \in \Phi_B$ we have, by Eqs. (A1) and (4.7),

$$B^*(H_\alpha) = \tfrac{1}{2}\langle H_\alpha, H_\alpha\rangle \alpha(B) = 0.$$

Thus on $\mathfrak{t}_{ss}$ the weight $\mu + B^*$ agrees with μ. The μ-dependent factor in the path-integral given by Eq. (4.6) factorizes into a semisimple piece and an Abelian piece. We want to show that shifting the θ by 2π is equivalent to multiplying the Abelian piece by

$$\exp\left(i \int B^*(A_0(t, 0))dt\right). \tag{4.10}$$

This is shown in the same way as in Sec. III. We note that the topological term in the action reduces to a boundary term of the form

$$S_\theta = -\frac{i\theta}{4\pi^2}\int_{r=\epsilon} \langle A_0, B_i n^i\rangle d^2\sigma dt,$$

where $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field, and the integration is over an $S^2 \times \mathbb{R} \subset \mathbb{R}^4$ given by the equation $r = \epsilon$. Taking into account the behavior of B_i for small r, performing the integral over S^2, and setting $\theta = 2\pi$, the above expression becomes

$$-i\int \langle A_0(t, 0), B\rangle dt.$$

Thus for $\theta = 2\pi$ the exponential of $-S_\theta$ is precisely given by Eq. (4.10), as claimed.

Finally, we would like to determine how S (or in the non-simply-laced case, ST^qS) acts on the pairs (μ, B). Since S-duality is still a conjecture, we have to guess the transformation law. Guided by the analogy with the Abelian case, we propose that S acts as follows:

$$S\colon (\mu, B) \mapsto (-B^*, \mu^*).$$

This transformation law has been previously considered in Ref. [18] in a somewhat different context. The same argument as above shows that $B^*(H_\alpha)$ is integral for all $\alpha \in \Phi$, so $-B^*$ is a weight of $\mathfrak{g}$. On the other hand, we have

$$\alpha(\mu^*) = \mu(\alpha^*) = \frac{2\mu(H_\alpha)}{\langle H_\alpha, H_\alpha\rangle}. \tag{4.11}$$

If $\mathfrak{g}$ is simply-laced, all coroots have length-squared equal to 2, so μ^* is a magnetic weight. But for non-simply-laced $\mathfrak{g}$ we also have longer coroots, so $\alpha(\mu^*)$ is not necessarily integral for all $\alpha \in \Phi$. Thus S as defined above is a well-defined operation on the set of Wilson-'t Hooft operators only in the simply-laced case. It is easy to check that C, T, S defined above satisfy the relations Eq. (4.8).

In the non-simply-laced case we consider a transformation ST^qS which acts as follows:

$$ST^qS\colon (\mu, B) \mapsto (-\mu, q\mu^* - B).$$

If we want this to be a legal transformation of Wilson-'t Hooft operators, then $q\alpha(\mu^*)$ must be an integer for all $\alpha \in \Phi$. According to Eq. (4.11), $\alpha(\mu^*)$ is an integer multiple of $1/2$ for $\mathfrak{g} = \mathsf{so}$, sp, and F_4, and an integer multiple of $1/3$ for $\mathfrak{g} = G_2$. Thus for $\mathfrak{g} = \mathsf{so}$, sp, and F_4 the largest possible duality group (among congruence subgroups of the form $\Gamma_0(q)$) is $\Gamma_0(2)$, while for $\mathfrak{g} = G_2$ it is $\Gamma_0(3)$. This agrees with Refs. [17,18].

E. BPS Wilson-'t Hooft operators

In the case of $N = 4$ SYM theory it is natural to consider line operators which preserve some supersymmetry. The simplest of these are $1/2$ BPS line operators. In the purely electric case, they are well known:

$$W_R^{\text{BPS}} = \text{Tr}_R P\exp\left(\int (iA_0(t, 0) + \phi(t, 0))dt\right),$$

where ϕ is one of the real scalars in the $N = 4$ multiplet, and R is an irreducible representation of G. They are sometimes called Maldacena-Wilson operators because of their role in AdS/CFT correspondence [26–28]. The scaling weight of the BPS Wilson operator at weak coupling is

$$h(W_R^{\text{BPS}}) = \frac{g^2 c_2(R)}{96\pi^2} + O(g^4). \tag{4.12}$$

Let us also define a BPS version of the 't Hooft operator. In addition to fixed boundary conditions for the gauge field, as in Eq. (4.2), we impose a fixed boundary condition for the scalar fields:

$$\phi^a = \frac{\lambda^a}{r} + O(1), \qquad a = 1, \ldots, 6,$$

where for each a λ^a is a covariantly constant section of the adjoint bundle on $S^2 \times \mathbb{R}$. These sections are determined by the BPS condition. To leading order in the gauge coupling, we may simply require that the solution of the classical equations of motion with the above asymptotics be BPS. This implies that

$$\lambda^a = \tfrac{1}{2} n^a B,$$

where n^a is a unit vector in $\mathbb{R}^6$. Using $SO(6)$ R-symmetry, we may always rotate n^a so that only one of its components is nonzero.

The general case of a BPS Wilson-'t Hooft operator is more complicated and will not be treated here fully. We only note that to leading order in g^2 one can simply neglect the "Wilson" part of the operator; then the behavior of the scalar field at $r = 0$ is the same as determined above.

F. Scaling weights of Wilson-'t Hooft operators in $N = 4$ super-Yang-Mills theory

In this subsection we compute the scaling weights of Wilson-'t Hooft operators (both BPS and non-BPS) in $N = 4$ super-Yang-Mills theory at weak coupling. Since properties of purely electric line operators (Wilson loops) are well-understood, we will focus on the case when the GNO "charge" $B \in \Lambda_{\mathrm{mw}}$ is nonzero. Recalling the computations in Sec. III, one can easily see that to leading order in the g^2 expansion one can neglect both the Wilson part of the operator, and the θ-term in the action. Therefore to this order it is sufficient to consider 't Hooft operators and set $\theta = 0$.

To evaluate the scaling weight to leading order in the weak-coupling expansion, we can simply evaluate the classical stress-energy tensor on the solution of classical equations of motion with the desired asymptotics at $r = 0$. This solution is

$$A_\mu dx^\mu = \frac{B}{2}(1 - \cos\theta)d\phi, \qquad \phi = \frac{\eta B}{2r}.$$

Here $\eta = 0$ corresponds to the ordinary 't Hooft operator (the S-dual of the ordinary Wilson operator), while $\eta = 1$ corresponds to the BPS 't Hooft operator. The bosonic part of the classical stress-energy tensor is

$$\begin{aligned} T_{\mu\nu} = {} & 2g^{-2}\,\mathrm{Tr}[D_\mu\phi D_\nu\phi - \tfrac{1}{2}\delta_{\mu\nu}(D\phi)^2 \\ & - \tfrac{1}{6}(D_\mu D_\nu - \delta_{\mu\nu}D^2)\phi^2] \\ & + 2g^{-2}\,\mathrm{Tr}[-F_{\mu\lambda}F_{\nu\lambda} + \tfrac{1}{4}\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}]. \end{aligned}$$

One easily finds that the scaling weight is

$$h(H_{B,\eta}) = \frac{1 - \frac{\eta^2}{3}}{4g^2}\langle B, B\rangle + O(1). \qquad (4.13)$$

S-duality predicts that the scaling weight of the 't Hooft operator at coupling g is equal to the scaling weight of the Wilson operator at coupling $\hat{g} = 4\pi/g$. Our weak-coupling results Eqs. (4.1), (4.12), and (4.13) show that this can be true only if the scaling weights of both Wilson and 't Hooft operators (whether BPS or not) receive higher-order corrections. Indeed, even the group-theory dependence of their scaling weights is very different at weak coupling. For example, for $G = SU(2)$ the scaling weight of the Wilson operator in the representation of isospin j goes like

$$h \sim j(j + 1),$$

while the scaling weight of the 't Hooft operator of "magnetic isospin" j goes like

$$h \sim j^2.$$

V. DISCUSSION AND OUTLOOK

In this paper we have studied line operators in 4d gauge theories which create electric and magnetic flux. In a free theory with gauge group $U(1)$, such operators are classified by a pair of integers, the electric and magnetic charges. Taking into account the results of Goddard, Nuyts, and Olive [10], one could guess that in the non-Abelian case Wilson-'t Hooft operators are classified by a pair of irreducible representations, one for the original gauge group G and the other for its magnetic dual $\hat{G}$. We will denote the set of irreducible representations of G by $\mathrm{IRep}(G)$.

With a little more thought, however, one realizes that in a non-Abelian theory there should be some interaction between the electric and magnetic representations. Our results show that such an interaction, although present, has a very simple form: instead of being labeled by a pair of irreducible representations, Wilson-'t Hooft operators are labeled by an element of $\Lambda_w \times \Lambda_{\mathrm{mw}}$, modulo the Weyl group of $\mathfrak{g}$. There is an obvious map from this set to the set $\mathrm{IRep}(G) \times \mathrm{IRep}(\hat{G})$, but this map is not injective. In other words, there is more information in a pair of weights modulo the action of the Weyl group than in the corresponding representation of $G \times \hat{G}$.

Let us illustrate this in the simple case $\mathfrak{g} = \mathsf{sl}_2$. Both lattices Λ_w and Λ_{mw} are one-dimensional in this case and can be identified with $\mathbb{Z}$. Thus a weight (either electric of magnetic) is simply an integer. The Weyl group is $\mathbb{Z}_2$, and its nontrivial element acts by negating both integers. An integer $m \in \mathbb{Z}$ corresponds to a representation of $SU(2)$ with isospin $j = |m|/2$. We see that if both electric and magnetic weights are nonzero, then there is precisely one more bit of information in the pair of weights than in the corresponding pair of representations. One can think of it as a "mutual orientation" of electric and magnetic representations.

This "interaction" between electric and magnetic representations makes it possible to have an action of the S-duality group on the set of Wilson-'t Hooft operators. Let us again illustrate our point in the example of $\mathfrak{g} = \mathsf{sl}_2$, where the S-duality group is $SL(2, \mathbb{Z})$. As explained above, one can index Wilson-'t Hooft operators by a pair of isospins j_e, j_m and an extra label $\xi \in \{1, -1\}$. The T and S transformations act as follows:

$$T\colon (j_e, j_m, \xi) \mapsto (|j_e + \xi j_m|, j_m, \mathrm{sign}(\xi j_e + j_m)),$$
$$S\colon (j_e, j_m, \xi) \mapsto (j_m, j_e, -\xi).$$

In contrast, no nontrivial $SL(2, \mathbb{Z})$ action is possible on the set of pairs of isospins.

In this paper we analyzed the action of S-duality on the Wilson-'t Hooft operators in $N = 4$ $d = 4$ SYM theory. It would be interesting to understand the action of other proposed dualities on line operators in $d = 4$ supersymmetric gauge theories. For example, it would be interesting to understand the action of dualities on line operators in finite $N = 2$ quiver theories. The conjectured duality group in this case is much more complicated than for $N = 4$ SYM: it is the fundamental group of the moduli space of flat irreducible connections on T^2, where the gauge group is simply-laced and determined by the type of the quiver [29,30]. One could also ask how the Wilson loop operator in $N = 1$ super-QCD transforms under Seiberg's duality [31]. Unlike in the $N = 4$ case, the physical origin of Seiberg's duality is obscure, and it is not clear whether Wilson operators are mapped to t' Hooft operators. A hint that this may indeed be so is provided by a work of M. Strassler [32], who argued that in $N = 1$ SUSY QCD with gauge group $SO(N)$ and vector matter Seiberg's duality maps Wilson operators corresponding to the spinor representation to line operators carrying nontrivial 't Hooft magnetic flux.

Another possible line of investigation is to study line operators in 3d theories. For example, one can "uplift" twist operators for free fermions and bosons in 2d to line operators creating topological disorder in the corresponding free 3d theories. A more nontrivial example is the "barbed-wire" line operator in the 3d Ising model defined by Dotsenko and Polyakov [33]. Recall that the 3d Ising model is related by Kramers-Wannier duality to a $\mathbb{Z}_2$ gauge theory. The most obvious line operator in this theory is the Wilson operator. The "barbed-wire" operator is obtained by decorating the Wilson operator with the Ising spin operators. Dotsenko and Polyakov showed that the "barbed-wire" operator satisfies a linear equation, which looks like a loop-space generalization of the Dirac equation. On the basis of this observation, they conjectured that the 3d Ising model may be integrable when expressed in terms of the "barbed-wire" operators. It would be interesting to test this conjecture by computing correlators of the "barbed-wire" operators with themselves and with local operators.

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APPENDIX: LIE ALGEBRA FACTS AND CONVENTIONS

In this appendix we record some basic definitions and conventions pertaining to compact simple Lie algebras and Lie groups. A standard reference on these matters is Ref. [34]. Let $\mathfrak{g}$ be such a Lie algebra. It has an Ad-invariant metric, which is unique up to a rescaling. We will not fix any particular normalization of the invariant metric on $\mathfrak{g}$, and therefore will not identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ (i.e. a maximal Abelian subalgebra of $\mathfrak{g}$), and let $\Phi \in \mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$. This means that $\mathfrak{g}_{\mathbb{C}}$ decomposes as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} V_\alpha,$$

such that for any $H \in \mathfrak{t}$ and any $X \in V_\alpha$ we have

$$[H, X] = \alpha(H)X.$$

The subspaces V_α are called root spaces; they can be shown to be one-dimensional. The span of Φ is the whole $\mathfrak{t}^*$.

It is always possible to choose a basis vector E_α for each V_α and a vector $H_\alpha \in \mathfrak{t}$ for each $\alpha \in \Phi$ so that

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \qquad [H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}.$$

The vector $E_\alpha \in V_\alpha$ is called the root vector corresponding to the root $\alpha \in \mathfrak{t}^*$, while $H_\alpha \in \mathfrak{t}$ is called the coroot corresponding to the root α. One can show that

$$\alpha(H_\beta) \in \mathbb{Z}, \qquad \forall \alpha, \qquad \beta \in \Phi.$$

The set of coroots spans $\mathfrak{t}$.

Using the restriction of the invariant metric to $\mathfrak{t}$, we can associate a vector $\alpha^* \in \mathfrak{t}$ to each root $\alpha \in \mathfrak{t}^*$. One can show that α^* is proportional to H_α, while their norms are related by

$$\langle \alpha^*, \alpha^* \rangle \cdot \langle H_\alpha, H_\alpha \rangle = 4.$$

This relation is independent of the particular normalization of the invariant metric. We will use this relation in the following form:

$$\alpha^* = \frac{2H_\alpha}{\langle H_\alpha, H_\alpha \rangle}, \qquad H_\alpha^* = \frac{1}{2}\alpha\langle H_\alpha, H_\alpha \rangle. \tag{A1}$$

Choosing a particular Cartan subalgebra breaks the gauge group down to a subgroup. The residual gauge transformations acting nontrivially on $\mathfrak{t}$ form a finite group $\mathcal{W}$ called the Weyl group of $\mathfrak{g}$. It consists of linear transformations w_α, $\alpha \in \Phi$, of the form

$$w_\alpha(H) = H - \alpha(H)H_\alpha, \qquad \forall H \in \mathfrak{t}.$$

These linear transformations are called Weyl reflections. The set of coroots is invariant with respect to the action of $\mathcal{W}$. The Weyl group also acts on the dual space $\mathfrak{t}^*$ as follows:

$$w_\alpha(f) = f - f(H_\alpha)\alpha, \qquad \forall f \in \mathfrak{t}^*.$$

The set of roots Φ is invariant with respect to this action.

The roots of $\mathfrak{g}$ span a lattice Λ_r in $\mathfrak{t}^*$ called the root lattice of $\mathfrak{g}$. Similarly, the coroots of $\mathfrak{g}$ span a lattice Λ_{cr} in $\mathfrak{t}$, called the coroot lattice. The dual of the coroot lattice is a lattice Λ_w in $\mathfrak{t}^*$ defined by the condition

$$f \in \Lambda_w \Leftrightarrow f(H_\alpha) \in \mathbb{Z}, \qquad \forall \alpha \in \Phi.$$

This lattice is called the weight lattice of $\mathfrak{g}$, and its elements are called weights of $\mathfrak{g}$. It is easy to see that the root lattice Λ_r is a sublattice of Λ_w. One can also show that the quotient lattice Λ_w/Λ_r is isomorphic to the center of $\tilde{G}$, the unique simply connected compact Lie group with Lie algebra $\mathfrak{g}$.

Dually, in $\mathfrak{t}$ we have a lattice $\Lambda_{\rm mw}$ defined by the condition

$$H \in \Lambda_{\rm mw} \Leftrightarrow \alpha(H) \in \mathbb{Z}, \qquad \forall \alpha \in \Phi.$$

This lattice is dual to the root lattice Λ_r and is called the lattice of magnetic weights of $\mathfrak{g}$ in the main text. The lattice Λ_{cr} is a sublattice of $\Lambda_{\rm mw}$, and their quotient is again the center of $\tilde{G}$.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. The kernel of the exponential mapping

$$\mathfrak{t} \rightarrow G, \qquad H \mapsto \exp(2\pi i H)$$

is a yet another lattice in $\mathfrak{t}$, which we call Γ_G. One has the inclusions

$$\Lambda_{\rm cr} \subset \Gamma_G \subset \Lambda_{\rm mw}.$$

The center and the fundamental group of G can be determined as follows:

$$Z(G) = \Lambda_{\rm mw}/\Gamma_G, \qquad \pi_1(G) = \Gamma_G/\Lambda_{\rm cr}.$$

The dual of Γ_G is a lattice in $\mathfrak{t}^*$ known as the weight lattice of G. We will denote it Γ_G^*. Obviously, we have inclusions

$$\Lambda_r \subset \Gamma_G^* \subset \Lambda_w$$

and group isomorphisms

$$Z(G) = \Gamma_G^*/\Lambda_r, \qquad \pi_1(G) = \Lambda_w/\Gamma_G^*.$$

A compact simple Lie algebra is called simply-laced if all its roots (i.e. all elements of Φ) have the same length, and non-simply-laced otherwise. The simply-laced Lie algebras are sl_N and so_{2N} series and the exceptional Lie algebras E_6, E_7, E_8. In the nonsimply-laced case, the roots can have only two different lengths, so one can meaningfully talk about short roots and long roots. The ratio of lengths-squared of long and short roots is either 2 (for Lie algebras sp_N, so_{2N+1}, and F_4) or 3 (for the exceptional Lie algebra G_2).

A compact simple Lie algebra can be reconstructed from its set of roots $\Phi \in \mathfrak{t}^*$ (provided that the invariant metric is also specified). A Lie algebra $\hat{\mathfrak{g}}$ is called the magnetic dual of $\mathfrak{g}$ if its set of roots coincides with the set of coroots of $\mathfrak{g}$. Each compact simple Lie algebra has a magnetic dual; applying the duality procedure twice gives back the Lie algebra one has started from. We also have the notion of a magnetic dual group $\hat{G}$: if G is a compact simple Lie group with Lie algebra $\mathfrak{g}$ and the kernel of the exponential mapping $\Gamma_G \subset \mathfrak{t}$, then the magnetic dual group $\hat{G}$ is uniquely defined by requiring that its Lie algebra be $\hat{\mathfrak{g}}$, and its kernel of the exponential mapping $\Gamma_{\hat{G}} \subset \hat{\mathfrak{t}} = \mathfrak{t}^*$ be the weight lattice Γ_G^*. In particular, if $G = \tilde{G}$, then the magnetic dual group has $\Gamma_{\hat{G}} = \Lambda_w$. By the above definitions, the lattice of magnetic weights for $\hat{\mathfrak{g}}$ is the weight lattice Λ_w for $\mathfrak{g}$, therefore the magnetic dual group $\hat{G}$ has a trivial center. In the main text such a group was denoted $\hat{G}_0$.

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